§6. Anomalies

§6.1 Transformation of the Measure (Abelian Anomaly)

For an arbitrary local trf.

$$\psi(x) \longmapsto \underset{\substack{\uparrow \\ \text{spin } 1/2 \text{ field}}}{} U(x)\psi(x)$$

the measure transforms as

$$[d\psi][d\bar{\psi}] \longmapsto (\operatorname{Det}\mathcal{U}\operatorname{Det}\bar{\mathcal{U}})^{-1}[d\psi][d\bar{\psi}]$$

where

$$\mathcal{U}_{xn,ym} \equiv U(x)_{nm}\,\delta^4(x-y),$$

$$\bar{\mathcal{U}}_{xn,ym} \equiv [\gamma_4 U(x)^\dagger \gamma_4]_{nm}\,\delta^4(x-y)$$

and $\gamma_4 \equiv i\gamma^0$ (from $\bar{\psi} = \psi^\dagger\gamma_4$)

n,m run over flavor labels and Dirac spin indices

Case 1:

$U(x)$ is unitary non-chiral transformation

$$U(x) = \exp[i\alpha(x)t],$$

with t Hermitian and $\alpha(x)$ real function.

$\rightarrow \quad \bar{\mathcal{U}}\,\mathcal{U} = \mathbb{1}$

hence $\operatorname{Det} \mathcal{U} = 1$

$\rightarrow$ measure remains invariant under this trf. (example: ordinary gauge trfs, $O(N)$ rotations etc.)

<u>Case II:</u>

Consider

$$\mathcal{U}(x) = \exp\left[i\gamma_5 \alpha(x)\, t\right]$$

with t and α as before.

$\rightarrow \quad \bar{\mathcal{U}} = \mathcal{U}$ (pseudo-Hermitian)

Thus measure is <u>not</u> invariant and we have

$$[d\psi][d\bar{\psi}] \mapsto (\operatorname{Det} \mathcal{U})^{-2} [d\psi][d\bar{\psi}]$$

Let us look at "infinitesimal" local trfs:

$$[\mathcal{U}-1]_{nx,my} = i\alpha(x)\,[\gamma_5 t]_{nm}\, \delta^4(x-y)$$

Using $\operatorname{Det} M = \exp \operatorname{Tr} \ln M$, $\ln(1+x) \rightarrow x$ for $x \to 0$

we get

$$(*) \quad [d\psi][d\bar{\psi}] \longrightarrow \exp\left\{ i \int d^4x\, \alpha(x)\, \mathcal{A}(x) \right\} [d\psi][d\bar{\psi}],$$

where

$$\mathcal{A} = -2\,\mathrm{Tr}\{\gamma_5 t\}\,\delta^4(x-x)$$

where "Tr" denotes a trace over both Dirac and flavor indices.

The factor $\exp\{i\int d^4x\, \alpha(x)\,\mathcal{A}(x)\}$ in the transformation rule (*) for the measure is equivalent to:

$$\mathcal{L}(x)_{\text{eff}} \longmapsto \mathcal{L}(x)_{\text{eff.}} + \alpha(x)\,\mathcal{A}(x)$$

$\rightarrow$ trf. property of effective Lagrangian where Fermions have been integrated out

Let's calculate $\mathcal{A}$!

$\rightarrow$ it's singular, so we have to regularize:

$$\mathcal{A}(x) = -2\left[\mathrm{Tr}\left\{\gamma_5\, t\, f(-\not{D}_x^2/M^2)\right\}\delta^4(x-y)\right]_{x\to y}$$

and $M \rightarrow \infty$ at the end s.t. $f(0)=1$

Properties of f:

$$f(0)=1,\quad f(\infty)=0,\quad \text{smooth}$$
$$s f'(s)=0 \quad \text{at} \quad s=0 \quad \text{and} \quad s=\infty \qquad (**)$$

D_x is Dirac diff. operator

$$(D_x)_\mu = \frac{\partial}{\partial x^\mu} - i t_\alpha A_{\alpha\mu}(x)$$

Using the Fourier-rep. of the δ-function we get

$$\mathcal{A}(x) = -2\int \frac{d^4k}{(2\pi)^4}\left[\mathrm{Tr}\left\{\gamma_5 t f(-\not{D}_x^2/M^2)\right\} e^{ik\cdot(x-y)}\right]_{y=x}$$

$$= -2\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left\{\gamma_5 t f(-[i\not{k} + \not{D}_x]^2/M^2)\right\}$$

Rescaling k^μ by a factor of M, this is

$$\mathcal{A}(x) = -2M^4\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left\{\gamma_5 t f(-[i\not{k} + \not{D}_x/M]^2)\right\}$$

Expanding the argument,

$$-\left[i\not{k} + \frac{\not{D}_x}{M}\right]^2 = k^2 - \frac{ik\cdot D_x}{M} - \left(\frac{\not{D}_x}{M}\right)^2$$

we see that in the limit $M\to\infty$ only 4 or less factors of $\frac{\not{D}_x}{M}$ survive

$$\to \mathcal{A}(x) = -\int \frac{d^4k}{(2\pi)^4} f''(k^2)\,\mathrm{Tr}\left\{\gamma_5 t \not{D}_x^4\right\}$$

traces over $\gamma_5 \not{D}_x^n$, $n<4$ vanish

→ independent of M

Performing a Wick-rotation $k^0 \mapsto ik^4$ with k^4 running from $-\infty$ to $+\infty$, we get

$$\int d^4k\, f''(k^2) = i\int_0^\infty 2\pi^2 k^3\, dk\, f''(k^2).$$

Using repeated partial integration and (**),
we get

$$\int d^4k\, f''(k^2) = i\pi^2 \int_0^\infty ds\, s\, f''(s) = -i\pi^2 \int_0^\infty ds\, f'(s) = i\pi^2$$

To calculate the trace, we write

$$\not{D}_x^2 = \frac{1}{4}\left\{(D_x)^\mu, (D_x)^\nu\right\}\left\{\gamma_\mu, \gamma_\nu\right\} + \frac{1}{4}\left[(D_x)^\mu, (D_x)^\nu\right]\left[\gamma_\mu, \gamma_\nu\right]$$

$$= D_x^2 - \frac{1}{4} i t_\alpha F_\alpha^{\mu\nu}\left[\gamma_\mu, \gamma_\nu\right]$$

Using

$$tr_D\left\{\gamma_5\left[\gamma_\mu, \gamma_\nu\right]\left[\gamma_\rho, \gamma_\sigma\right]\right\} = 16 i \varepsilon_{\mu\nu\rho\sigma}$$

gives us

$$\mathcal{A}(x) = -\frac{1}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma} F_\alpha^{\mu\nu}(x) F_\beta^{\rho\sigma}(x)\, tr\{t_\alpha t_\beta t\},$$

where "tr" here runs only over the flavor indices

Let's apply this to Pion-decay!

It is observed that π^0 (neutral isospin pion) decays as

(2) $\quad \pi^0 \longrightarrow 2\gamma\,, \quad \mathcal{L}_{\pi\gamma\gamma} = g\pi^0 \varepsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}$

Predicted decay rate:

$$\Gamma(\pi^0 \to 2\gamma) = \frac{m_\pi^3 g^2}{\pi}\,, \quad g = \frac{e^2}{8\pi^2 F_\pi}\left(\frac{m_\pi^2}{m_N^2}\right)$$

→ incompatible with experiment (too low!)

Now let's compute it using our anomaly $\mathcal{A}$:

charge-neutral chiral trf:

$$\delta u = i\alpha\gamma_5 u, \qquad \delta d = -i\alpha\gamma_5 d \qquad (1)$$

→ symmetry is "anomalous" in the presence of electromagnetic field:

$$\mathcal{A}(x) = -\frac{1}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}(x)F^{\rho\sigma}(x)\,\mathrm{tr}\{q^2\tau_3\}$$

with q the quark charge matrix and

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \left(\text{acting on } \begin{pmatrix} u \\ d \end{pmatrix}\right)$$

Have - N_c u quarks of charge $\frac{2}{3}e$

- N_c d quarks of charge $-\frac{e}{3}$

$$\rightarrow \mathrm{tr}\{q^2\tau_3\} = N_c\left(\frac{2e}{3}\right)^2(+1) + N_c\left(\frac{-e}{3}\right)^2(-1) = \frac{N_c e^2}{3},$$

giving

$$\mathcal{A}(x) = -\frac{N_c e^2}{48\pi^2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}(x)F^{\rho\sigma}(x)$$

→ must include terms in Lagrangian such that under (1) we have anomalous trf.

$$\delta\mathcal{L}_{eff}(x) = \alpha\mathcal{A}(x) = -\frac{N_c e^2}{48\pi^2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}(x)F^{\rho\sigma}(x)\alpha$$

recall under (1) (broken symmetry)

$$\delta \pi^0 = 2\alpha \langle \sigma \rangle = \alpha F_\pi \quad , \quad F_\pi = 184 \text{ MeV}$$

$\rightarrow$ include in effective Lagrangian:

$$\frac{\pi^0(x)\, \mathcal{A}(x)}{F_\pi} = \frac{-N_c e^2}{48\pi^2 F_\pi} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu}(x) F^{\rho\sigma}(x)\, \pi^0(x)$$

Comparing with (2), we get for g:

$$g = \frac{N_c e^2}{48\pi^2 F_\pi}$$

$$\rightarrow \Gamma(\pi^0 \rightarrow 2\gamma) = \frac{N_c^2 \alpha^2 m_\pi^3}{144\pi^3 F_\pi^2} = \left(\frac{N_c}{3}\right)^2 \times 1.11 \times 10^{16} s^{-1}$$

Observed rate is $\Gamma(\pi^0 \rightarrow 2\gamma) = (1.19 \pm 0.08) \times 10^{16} s^{-1}$

in good agreement iff $N_c = 3$!

$\rightarrow$ evidence that there are 3 quark colors!